ELASTICITY-THEORETIC PLANE PROBLEM FOR AN ANISOTROPIC BODY SUBJECT TO AN ELECTRICAL EFFECT

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The elastic properties of the body are assumed inseparable from its electrical properties (piezoelectric effect). The static problem is considered for an infinitely long homogeneous rod under the action of a plane system of forces or a lengthwise uniform distribution of electric field potential. The general equations of theory of elasticity and of electrical field theory, together with the piezoelectric relationships, are used. Conditions on the material parameters are obtained, under which a plane electric field and a plane deformation state appear in the body.

The stressed state of a continuous body is characterized [1] by six stress components σ_{ij} (i, j = 1, 2, 3), taken in an orthogonal rectilinear coordinate system x_i (i = 1, 2, 3), the orientation of which, relative to the crystallographic coordinate system, is precisely known. In the absence of spatial forces, the stresses have to satisfy the equilibrium conditions

$$\partial\sigma_{ij}/\partial x_j = 0 \tag{1}$$

In the case of small deformations, the components ξ_{ij} (i, j = 1, 2, 3) of the relative deformation are connected with the projections of the point displacements u_i (i = 1, 2, 3) by

$$\xi_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \quad (i \neq j), \qquad \xi_{ij} = \frac{\partial u_i}{\partial x_j} \quad (i = j) \tag{2}$$

In the absence of spatial charge [2], the components of the electric induction vector D_i (i = 1, 2, 3) satisfy the Maxwell's equation

$$\partial D_i \,/\, \partial x_i = 0 \tag{3}$$

while the electric field potential U (a scalar function) is connected with the components of the electric field-strength vector $\vartheta_i \,(i = 1, 2, 3)$ by

$$\vartheta_i = -\frac{\partial U}{\partial x_i} \tag{4}$$

The generalized Hooke's law does not hold in this case. An even more general law has to be applied, linearly connecting the electrical as well as the mechanical quantities. One form of it is

$$D_i = \varepsilon_{ij}^{\xi}\vartheta_j - e_{ikl}\xi_{kl}, \quad \sigma_{kl} = e_{ikl}\vartheta_i + c_{klpq}^{\vartheta}\xi_{pq} \tag{5}$$
$$(i,\ j,\ k,\ l,\ p,\ q = 1,\ 2,\ 3)$$

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Here, ε_{ij}^{ξ} is the dielectric constant of the crystal in the absence of deformations, c_{klpq}^{ϑ} are the coefficients of elasticity in the absence of an electric field, and e_{ikl} are the piezoelectric moduli. The values of these coefficients depend on both the material and the chosen coordinate system; whatever the coodrinate system, the tensors

$$\varepsilon_{ij} = \varepsilon_{ji}, \quad c_{klpq} = c_{lkpq} = c_{klqp} = c_{lkqp}, \quad e_{ikl} = e_{ilk}$$

are symmetric.

If the material is inhomogeneous, the tensors ε, c, and e are functions of the coordinates. Here, a homogeneous material is assumed, i.e., the tensor coefficients are constants.

Given an infinitely long cylinder with generator parallel to the x_3 axis. Since the electrical boundary conditions are essentially dependent on the shape of the body, the cross section of the cylinder by a plane perpendicular to its generator will be assumed to be a rectangle. The problem to be considered is that in which the direct piezoelectric effect is utilized. The external forces applied to the cylinder surface will be assumed normal to the generators and uniformly distributed along a generator (the so-called plane system of forces, usually employed in theory of elasticity).

In the case of an anisotropic (as distinct from isotropic) body, this system of forces is well known [1] not to give rise in general to a plane deformation state. In order for the deformation to be plane, restrictions have to be imposed on the material properties. These restrictions were obtained by S. G. Mikhlin [4] for the general case of an anisotropic material in the absence of electrical effects. The aim here is to find the conditions to be satisfied by the coefficients of elasticity, the components of the dielectric constant, and the piezoelectric moduli, in order for a plane electric field and plane deformation state to arise in the body under the action of a plane system of forces satisfying the conditions of statics.

In short, let

$$\partial u_1 / \partial x_3 = 0, \quad \partial u_2 / \partial x_3 = 0, \quad u_3 = 0, \quad \partial U / \partial x_3 = 0 \tag{6}$$

Then, from Eqs. (2) and (4),

$$\vartheta_3 = 0, \qquad \xi_{33} = \xi_{23} = \xi_{13} = 0 \tag{7}$$

while Eq. (5) takes the simplified form

$$\begin{aligned} D_i &= \varepsilon_{ij}^{\xi}\vartheta_j - e_{ikl}\xi_{kl} \\ \sigma_{pq} &= e_{jpq}\vartheta_j + c_{pqkl}^{\vartheta}\xi_{kl} \end{aligned} \quad \begin{pmatrix} i, p, q = 1, 2, 3 \\ j, k, l = 1, 2 \end{pmatrix} \tag{8}$$

From (8), all the D_i and σ_{pq} are independent of x_3. Hence, Eqs. (1) and (3) become

$$\frac{\partial D_1}{\partial x_1} + \frac{\partial D_2}{\partial x_2} = 0 \tag{9}$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0, \quad \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0, \quad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0 \tag{10}$$

Let φ, ψ, and θ be new unknown functions, defined so as to satisfy Eqs. (9) and (10):

$$D_1 = \frac{\partial \varphi}{\partial x_2}, \qquad D_2 = -\frac{\partial \varphi}{\partial x_1} \tag{11}$$

$$\sigma_{11} = \frac{\partial^2 \psi}{\partial x_2^2}, \qquad \sigma_{22} = \frac{\partial^2 \psi}{\partial x_1^2}, \qquad \sigma_{12} = -\frac{\partial^2 \psi}{\partial x_1 \partial x_2} \tag{12}$$

$$\sigma_{23} = \frac{\partial \theta}{\partial x_1}, \qquad \sigma_{13} = -\frac{\partial \theta}{\partial x_2} \tag{13}$$

Let F denote the 5-dimensional column vector

$$F = \| D_i, \ \sigma_{kl} \|^T \quad (i, k, l = 1, 2; \ k \leqslant l) \tag{14}$$

and E the row vector

$$E = \| \vartheta_i, \ \xi_{kl} \| \tag{15}$$

while A is the 5×5 square matrix of coefficients

$$A = \left\| \begin{array}{c|c} \varepsilon_{ij}^{\xi} & -e_{ikl} \\ \hline e_{ikl} & c_{klpq}^{\vartheta} \end{array} \right\| \qquad \begin{pmatrix} i, j, k, l, p, q = 1, 2 \\ k \leqslant l, \quad p \leqslant q \end{pmatrix} \tag{16}$$

The relevant part of Eq. (8) can then be written briefly in the matrix form

$$F = AE \tag{17}$$

It is always possible to solve Eq. (17) for E:

$$E = BF \tag{18}$$

where the inverse $B = A^{-1}$ of A is asymmetric like A. The notation for the remaining part of system (8) is

$$F_3 = A'E \tag{19}$$

$$F_3 = \| D_3, \sigma_{33}, \sigma_{23}, \sigma_{13} \|^T \tag{20}$$

$$A' = \left\| \begin{array}{c|c} \varepsilon_{3i}^{\xi} & -e_{3jk} \\ \hline e_{il3} & c_{l3jk}^{\vartheta} \end{array} \right\| \qquad \begin{pmatrix} i, j, k = 1, 2 \\ l = 1, 2, 3 \end{pmatrix} \tag{21}$$

Here, A' is a 4×5 matrix with elements $\| a_{pq}' \|$, and the subscript 3 in Eq. (19) indicates that 3 is present in the subscript of each component of the vector F_3.

To transform Eq. (18) in such a way that its left side becomes identically zero, Eqs. (11) and (12) are utilized on its right side, and Eqs. (2) and (4) on its left. After manipulations such as taking the partial derivatives and adding equations, a system of two differential equations in the two unknown functions φ and ψ is obtained:

$$L_4\psi - L_3\varphi = 0, \qquad L_3\psi + L_2\varphi = 0 \tag{22}$$

Here, L_2, L_3, and L_4 are partial differential operators with constant coefficients, of orders 2, 3, and 4, respectively:

$$\begin{aligned} L_2 &= b_{22}\frac{\partial^2}{\partial x_1^2} - 2b_{12}\frac{\partial^2}{\partial x_1 \partial x_2} + b_{11}\frac{\partial^2}{\partial x_2^2} \\ L_3 &= -b_{24}\frac{\partial^3}{\partial x_1^3} + (b_{14} + b_{25})\frac{\partial^3}{\partial x_1^2 \partial x_2} - (b_{15} + b_{23})\frac{\partial^3}{\partial x_1 \partial x_2^2} + b_{13}\frac{\partial x^3}{\partial x_2^3} \\ L_4 &= b_{44}\frac{\partial^4}{\partial x_1^4} - 2b_{45}\frac{\partial^4}{\partial x_1^3 \partial x_2} + (2b_{34} + b_{55})\frac{\partial^4}{\partial x_1^2 \partial x_2^2} - 2b_{35}\frac{\partial^4}{\partial x_1 \partial x_2^3} + b_{33}\frac{\partial^4}{\partial x_2^4} \end{aligned} \tag{23}$$

All the unknown functions will be determined below by means of these equations. But φ and ψ satisfy a further equation, independent of those so far utilized. Using the relevant equations of Eq. (8), the third equation of Eq. (10) gives

$$e_{ik3}\frac{\partial \vartheta_i}{\partial x_k} + c_{i3kl}^{\vartheta}\frac{\partial \xi_{kl}}{\partial x_i} = 0 \qquad (i, k, l = 1, 2)$$

Replace ϑ_1, ϑ_2, ξ_{11}, ξ_{22}, ξ_{12} by their expressions from (18):

$$\left(\alpha_1\frac{\partial^2}{\partial x_1^2} + \alpha_2\frac{\partial^2}{\partial x_1 \partial x_2} + \alpha_3\frac{\partial^2}{\partial x_2^2}\right)\varphi + \left(\alpha_4\frac{\partial^3}{\partial x_1^3} + \alpha_5\frac{\partial^3}{\partial x_1^2 \partial x_2} + \alpha_6\frac{\partial^3}{\partial x_1 \partial x_2^2} + \alpha\frac{\partial^3}{\partial x_2^3}\right)\psi = 0 \tag{24}$$

Here,

$$\begin{aligned} \alpha_1 &= -(e_{113}b_{12} + e_{223}b_{22} + c_{1311}^{\vartheta}b_{32} + c_{1322}^{\vartheta}b_{42} + c_{1312}^{\vartheta}b_{52}) \\ \alpha_2 &= e_{123}b_{11} + e_{223}b_{21} + c_{1311}^{\vartheta}b_{31} + c_{1322}^{\vartheta}b_{41} + c_{1312}^{\vartheta}b_{51} - (e_{123}b_{12} + e_{223}b_{22} + c_{2311}^{\vartheta}b_{32} + c_{2322}^{\vartheta}b_{42} + c_{2312}^{\vartheta}b_{52}) \\ \alpha_3 &= e_{123}b_{11} + e_{223}b_{21} + c_{1311}^{\vartheta}b_{31} + c_{2322}^{\vartheta}b_{41} + c_{2312}^{\vartheta}b_{51} \\ \alpha_4 &= e_{113}b_{14} + e_{213}b_{24} + c_{1311}^{\vartheta}b_{34} + c_{1322}^{\vartheta}b_{44} + c_{1312}^{\vartheta}b_{54} \\ \alpha_5 &= e_{123}b_{14} + e_{223}b_{24} + c_{2311}^{\vartheta}b_{34} + c_{2322}^{\vartheta}b_{44} + c_{2312}^{\vartheta}b_{54} - (e_{113}b_{15} + e_{213}b_{25} + c_{1311}^{\vartheta}b_{35} + c_{1322}^{\vartheta}b_{45} + c_{1312}^{\vartheta}b_{55}) \\ \alpha_6 &= e_{113}b_{13} + e_{213}b_{23} + c_{1311}^{\vartheta}b_{33} + c_{1322}^{\vartheta}b_{43} + c_{1312}^{\vartheta}b_{53} - (e_{123}b_{15} + e_{223}b_{25} + c_{2311}^{\vartheta}b_{35} + c_{2322}^{\vartheta}b_{45} + c_{2312}^{\vartheta}b_{55}) \\ \alpha_7 &= e_{123}b_{13} + e_{223}b_{23} + c_{2311}^{\vartheta}b_{33} + c_{2322}^{\vartheta}b_{43} + c_{2312}^{\vartheta}b_{53} \end{aligned} \tag{25}$$

Every solution of Eq. (22) must also satisfy (24). In [4] it is shown that this happens if and only if all the coefficients of Eq. (24) are zero. Equating all the $\alpha_i (i = 1, 2, ..., 7)$ to zero, the required conditions are obtained, under which the action of a plane system of forces gives rise to a plane electric field and plane deformation state in the cylinder:

$$\alpha_i = 0 \qquad (i = 1, 2, ..., 7) \tag{26}$$

An especially simple particular case of (26) is

$$\begin{gathered} c^{э}_{1123} = c^{э}_{2223} = c^{э}_{1113} = c^{э}_{2213} = c^{э}_{2312} = c^{э}_{1222} = 0 \\ e_{123} = e_{223} = e_{113} = e_{213} = 0 \end{gathered} \tag{27}$$

Condition (27) is satisfied by crystals belonging [3] to the monoclinic antihemihedral class of the monoclinic system (if the plane of symmetry is at right angles to the z axis), or to the trigonal-dipyramidal class of the hexagonal system or ditrigonal-pyramidal class of the same system (if the plane of symmetry is at right angles to the x or y axis).

The rod must be cut in such a way that its ribs are parallel to the crystallographic coordinate system, and x_3 axis perpendicular to the plane of symmetry, if there is one. Condition (27) is also satisfied by SbSI crystals obtained by spontaneous polarization of the symmetry group, corresponding to the rhombo-pyramidal class of the rhombic system. Our list of examples is not meant to be exhaustive; it merely indicates the practicability of the assumptions made, such as Eqs. (6) and (7).

Returning to Eq. (22), general expressions will be found for φ and ψ. Denote by $P_6(D)$ the square matrix of differential operators of Eq. (22). The necessary condition for Eq. (22) to have nontrivial solution is

$$\det P_6 (\lambda) = 0 \tag{28}$$

Both unknowns satisfy the same sixth-order differential equation, e.g.,

$$P_6 (D)\varphi = 0 \tag{29}$$

The general solution for functions φ and ψ will obviously be the same. Condition (28) can be written in the more detailed form

$$l_2 (\lambda) l_4 (\lambda) + l_3^2 (\lambda) = 0 \tag{30}$$

where

$$\begin{aligned} l_2 (\lambda) &= b_{11}\lambda^2 - 2b_{12}\lambda + b_{22} \\ l_3 (\lambda) &= b_{13}\lambda^3 - (b_{15} + b_{23})\lambda^2 + (b_{14} + b_{25})\lambda - b_{24} \\ l_4 (\lambda) &= b_{33}\lambda^4 - 2b_{35}\lambda^3 + (2b_{34} + b_{55})\lambda^2 - 2b_{45}\lambda + b_{14} \end{aligned} \tag{31}$$

Equation (29) can be written as

$$D_6 D_5 D_4 D_3 D_2 D_1 \varphi = 0 \tag{32}$$

$$D_i \equiv \frac{\partial}{\partial x_2} - \lambda_i \frac{\partial}{\partial x_1} \qquad (i = 1, 2, \ldots, 6) \tag{33}$$

where D_i are linear first-order operators, and λ_i the roots of (30).

The general solution of (32) may be found by the method of successive integration described in [5] and used by S. G. Lekhnitskii in [1]. Let (30) have no multiple roots, and denote

$$D_1 \varphi = z_2, \quad D_2 z_2 = z_3, \quad \ldots, \quad D_5 z_5 = z_6 \tag{34}$$

The function z_6 satisfies the first order equation

$$D_6 z_6 = 0 \tag{35}$$

which after changing the variables as follows:

$$z_6 (x_1, x_2) = \omega (\eta_1, \eta_2), \quad x_1 + \lambda_6 x_2 = \eta_1, \quad x_1 - \lambda_6 x_2 = \eta_2$$

can be written as

$$2\frac{\partial\omega}{\partial\eta_1}=0$$

The general solution of this equation is $\omega = F(\eta_1)$. Hence, $z_6 = F(x_1 + \lambda_6 x_2)$. The function F may be written as the fifth derivative of another arbitrary function with respect to the argument $(x_1 + \lambda_6 x_2)$:

$$z_6 = f_6^{(V)}(x_1+\lambda_6 x_2) \tag{36}$$

The function z_5 satisfies the inhomogeneous first-order equation

$$D_5 z_5 = f_6^{(V)}(x_1+\lambda_6 x_2) \tag{37}$$

whose general solution is the sum of the general solution of

$$D_5 z_5 = 0 \tag{38}$$

and a particular solution of the inhomogeneous equation. Write the general solution of (38) as the fourth derivative of an arbitrary function $f_5(x_1 + \lambda_5 x_2)$ with respect to its argument $(x_1 + \lambda_5 x_2)$. The particular solution $z_5 = z_5(x_1 + \lambda_6 x_2)$ must satisfy (37). Taking

$$\frac{\partial}{\partial x_1} z_5 = z_5{}', \qquad \frac{\partial}{\partial x_2} z_5 = \lambda_6 z_5{}'$$

and substituting in (37),

$$\lambda_6 z_5{}' - \lambda_5 z_5{}' = f^{(V)}{}_6$$

Hence,

$$z_5 = \frac{f_6^{(IV)}}{\lambda_6-\lambda_5}$$

The general solution of (37) is now

$$z_5 = f_5^{(IV)}(x_1+\lambda_5 x_2) + \frac{f_6^{(IV)}(x_1+\lambda_6 x_2)}{\lambda_6-\lambda_5} \tag{39}$$

Proceeding with similar operations, the following expressions are obtained for φ and ψ:

$$\varphi = \sum_{k=1}^{6} \varphi_k(x_1+\lambda_k x_2) \tag{40}$$

$$\psi = \sum_{k=1}^{6} \psi_k(x_1+\lambda_k x_2) \tag{41}$$

Here we have written

$$\varphi_1(x_1+\lambda_1 x_2) = f_1(x_1+\lambda_1 x_2)$$
$$\varphi_k(x_1+\lambda_k x_2) = \frac{f_k(x_1+\lambda_k x_2)}{(\lambda_k-\lambda_1)\dots(\lambda_k-\lambda_{k-1})}$$

and similarly for the ψ_k, where φ_k and ψ_k are arbitrary functions of the argument $(x_1 + \lambda_k x_2)$, with derivatives up to and including order six with respect to their argument. The general expressions for the unknowns φ and ψ, which are real functions of x_1 and x_2, are

$$\begin{aligned}\varphi &= \sum_{k=1}^{3}[\varphi_k(y_k)+\bar{\varphi}_k(\bar{y}_k)] = 2\,\mathrm{Re}\sum_{k=1}^{3}\varphi_k(y_k)\\ \psi &= \sum_{k=1}^{3}[\psi_k(y_k)+\bar{\psi}_k(\bar{y}_k)] = 2\,\mathrm{Re}\sum_{k=1}^{3}\psi_k(y_k)\end{aligned} \tag{42}$$

where

$$y_k = x_1 + \lambda_k x_2, \ \bar{y}_k = x_1 + \bar{\lambda}_k x_2$$
$$\lambda_k = \alpha_k + i\beta_k, \ \lambda_{k+3} = \bar{\lambda}_k = \alpha_k - i\beta_k \ (k = 1, 2, 3; \ \beta_k > 0)$$

and $\bar{\varphi}_k(\bar{y}_k)$, $\bar{\psi}_k(\bar{y}_k)$ are the conjugate functions to $\varphi_k(y_k)$ and $\psi_k(y_k)$.

It can be shown by using energy considerations and Lekhnitskii's method [1] that Eq. (30) has no real roots. It must be borne in mind here that the potential energy W per unit volume of the crystal is here given by

$$2W = c_{klpq}^{\vartheta}\xi_{kl}\xi_{pq} - \varepsilon_{ij}^{\xi}\vartheta_i\vartheta_j + 2e_{ikl}\vartheta_i\xi_{kl} \tag{43}$$

which is positive. It will be assumed henceforth that all the roots of Eq. (30) are simple and are not simultaneously roots of the equations

$$l_4(\lambda) = 0, \qquad l_3(\lambda) = 0, \qquad l_2(\lambda) = 0$$

Denoting

$$\psi_1'(y_1) = \Phi_1(y_1), \qquad \psi_2'(y_2) = \Phi_2(y_2), \qquad \varphi_3(y_3) = \Phi_3(y_3)$$

all the unknowns of the problem may be expressed in terms of these three functions. Notice that

$$\begin{gathered}
\frac{\partial\psi}{\partial x_1} = 2\,\mathrm{Re}\,[\Phi_1(y_1) + \Phi_2(y_2) + h_3\Phi_3(y_3)] \\
\frac{\partial\psi}{\partial x_2} = 2\,\mathrm{Re}\,[\lambda_1\Phi_1(y_1) + \lambda_2\Phi_2(y_2) + \lambda_3 h_3\Phi_3(y_3)] \\
\varphi = 2\,\mathrm{Re}\,[h_1\Phi_1(y_1) + h_2\Phi_2(y_2) + \Phi_3(y_3)] \\
h_1 = -\frac{l_3(\lambda_1)}{l_2(\lambda_1)}, \qquad h_2 = -\frac{l_3(\lambda_2)}{l_2(\lambda_2)}, \qquad h_3 = \frac{l_3(\lambda_3)}{l_4(\lambda_3)}
\end{gathered} \tag{44}$$

Substituting Eq. (44) in (17) and recalling Eqs. (11) and (12), the vector F may be found by a single differentiation:

$$F_i = 2\,\mathrm{Re}\sum_{j=1}^{3}\alpha_{ij}\Phi_j'(y_j) \qquad (i = 1, 2, \ldots, 5) \tag{45}$$

where

$$\begin{gathered}
\alpha_{11} = h_1\lambda_1, \quad \alpha_{12} = h_2\lambda_2, \quad \alpha_{13} = \lambda_3 \\
\alpha_{21} = -h_1, \quad \alpha_{22} = -h_2, \quad \alpha_{23} = -1 \\
\alpha_{31} = \lambda_1^2, \quad \alpha_{32} = \lambda_2^2, \quad \alpha_{33} = \lambda_3^2 h_3 \\
\alpha_{41} = 1, \quad \alpha_{42} = 1, \quad \alpha_{43} = h_3 \\
\alpha_{51} = -\lambda_1, \quad \alpha_{52} = -\lambda_2, \quad \alpha_{53} = -\lambda_3 h_3
\end{gathered}$$

From (18),

$$E_i = 2\,\mathrm{Re}\sum_{j=1}^{3}\beta_{ij}\Phi_j'(y_j) \tag{46}$$

where

$$\beta_{ij} = b_{ik}\alpha_{kj} \qquad (i, k = 1, 2, \ldots, 5)$$

From Eq. (19), the vector F_3 has the components

$$F_{3p} = 2\,\mathrm{Re}\sum_{j=1}^{3}\gamma_{pj}\Phi_j'(y_j) \tag{47}$$

where

$$\gamma_{pj} = a_{pq}'\beta_{qj} \qquad (p = 1, \ldots, 4;\ q = 1, \ldots, 5)$$

Integration of (46) gives the expressions for the electric field potential U and the projections of the displacement vector u_i (i = 1, 2, 3):

$$U(x_1, x_2) = U_0 - 2\,\mathrm{Re}\left[\sum_{j=1}^{3}\left(\beta_{1j} + \frac{\beta_{2j}}{\lambda_j}\right)\Phi_j(y_j)\right] \tag{48}$$

$$\begin{aligned} u_1(x_1, x_2) &= u_{10} - \omega x_2 + 2\,\mathrm{Re}\left[\sum_{j=1}^{3}\left(\beta_{3j} + \frac{\beta_{5j}}{2\lambda_j}\right)\Phi_j(y_j)\right] \\ u_2(x_1, x_2) &= u_{20} + \omega x_1 + 2\,\mathrm{Re}\left[\sum_{j=1}^{3}\left(\frac{\beta_{4j}}{\lambda_j} + \frac{\beta_{5j}}{2}\right)\Phi_j(y_j)\right] \end{aligned} \tag{49}$$

where U_0, u_{10}, u_{20}, ω are constants of integration, indicating the zero level of the electric field potential and the "rigid" displacement of the crystal as a whole.

Consider the possible boundary conditions to be satisfied by the functions $\Phi_j(y_j)$ (j = 1, 2, 3).

The Direct Piezoelectric Effect. Here, σ_{1n} and σ_{2n} are given on the contour of the cross section, and $D_n = 0$ [6]. Then

$$\begin{aligned} 2\mathrm{Re}\,[\Phi_1(y_1) + \Phi_2(y_2) + h_3\Phi_3(y_3)] &= -\frac{1}{2}\int_0^s \sigma_{2n}\,ds + C_2 \\ 2\mathrm{Re}\,[\lambda_1\Phi_1(y_1) + \lambda_2\Phi_2(y_2) + \lambda_3 h_3\Phi_3(y_3)] &= \frac{1}{2}\int_0^s \sigma_{1n}\,ds + C_1 \\ 2\,\mathrm{Re}\,[h_1\Phi_1(y_1) + h_2\Phi_2(y_2) + \Phi_3(y_3)] &= C_3 \end{aligned} \tag{50}$$

where s is the arc along the contour, measured from some fixed origin s = 0; n is the outward normal to the contour; and C_1, C_2, and C_3 are constants, which can be assumed zero for a simply connected region.

Given the displacements u^*_1 and u^*_2 of points of the contour in the plane of cross section, while D_n = 0. The functions $\Phi_j(y_j)$ (j = 1, 2, 3) then satisfy

$$\begin{aligned} 2\mathrm{Re}\left[\sum_{j=1}^{3}\left(\beta_{3j} + \frac{\beta_{5j}}{2\lambda_j}\right)\Phi_j(y_j)\right] &= u_1^* - u_{10} + \omega x_2 \\ 2\mathrm{Re}\left[\sum_{j=1}^{3}\left(\frac{\beta_{4j}}{\lambda_j} + \frac{\beta_{5j}}{2}\right)\Phi_j(y_j)\right] &= u_2^* - u_{20} - \omega x_1 \\ 2\mathrm{Re}\,[h_1\Phi_1(y_1) + h_2\Phi_2(y_2) + \Phi_3(y_3)] &= C \end{aligned} \tag{51}$$

The Inverse Piezoelectric Effect. Given the electric field potential U* on the contour and the conditions $u^*_1 = 0$, $u^*_2 = 0$ for a mechanically clamped crystal,

$$\begin{aligned} 2\mathrm{Re}\left[\sum_{j=1}^{3}\left(\beta_{1j} + \frac{\beta_{2j}}{\lambda_j}\right)\Phi_j(y_j)\right] &= U_0 - U^* \\ 2\mathrm{Re}\left[\sum_{j=1}^{3}\left(\beta_{3j} + \frac{\beta_{5j}}{2\lambda_j}\right)\Phi_j(y_j)\right] &= -u_{10} + \omega x_2 \\ 2\mathrm{Re}\left[\sum_{j=1}^{3}\left(\frac{\beta_{4j}}{\lambda_j} + \frac{\beta_{5j}}{2}\right)\Phi_j(y_j)\right] &= -u_{20} - \omega x_1 \end{aligned} \tag{52}$$

Given the potential U^* and the conditions $\sigma_{1n} = 0$ and $\sigma_{2n} = 0$ for a mechanically unconstrained crystal,

$$\begin{aligned} 2\mathrm{Re}\left[\sum_{j=1}^{3}\left(\beta_{1j} + \frac{\beta_{2j}}{\lambda_j}\right)\Phi_j(y_j)\right] &= U_0 - U^* \\ 2\mathrm{Re}\,[\Phi_1(y_1) + \Phi_2(y_2) + h_3\Phi_3(y_3)] &= C_2 \\ 2\mathrm{Re}\,[\lambda_1\Phi_1(y_1) + \lambda_2\Phi_2(y_2) + \lambda_3 h_3\Phi_3(y_3)] &= C_1 \end{aligned}$$

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